Angle Chasing

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1 Facts you should know

1. Let ABC be a triangle and extend BC past C to D. Show that $\angle ACD = \angle BAC + \angle ABC$.

2. Let ABC be a triangle with $\angle C = 90$. Show that the circumcenter is the midpoint of AB.

3. Let ABC be a triangle with orthocenter H and feet of the altitudes D, E, F. Prove that H is the incenter of $\triangle DEF$.

4. Let ABC be a triangle with orthocenter H and feet of the altitudes D, E, F. Prove (i) that A, E, F, H lie on a circle diameter AH and (ii) that B, E, F, C lie on a circle with diameter BC.

5. Let ABC be a triangle with circumcenter O and orthocenter H. Show that $\angle BAH = \angle CAO$.

6. Let ABC be a triangle with circumcenter O and orthocenter H and let AH and AO meet the circumcircle at D and E, respectively. Show (i) that H and D are symmetric with respect to BC, and (ii) that H and E are symmetric with respect to the midpoint BC.

7. Let ABC be a triangle with altitudes AD, BE, and CF. Let M be the midpoint of side BC. Show that ME and MF are tangent to the circumcircle of AEF.

8. Let ABC be a triangle with incenter I, A-excenter I_a, and D the midpoint of arc BC not containing A on the circumcircle. Show that $DI = DI_a = DB = DC$.

9. Let ABC be a triangle with incenter I and D the midpoint of arc BC not containing A on the circumcircle. Define E and F similarly. Show (i) that I is the orthocenter of DEF, and (ii) that A, B, C are the reflections of I across EF, FD, DE respectively.

10. Let ABC be a triangle with incenter I and excenters I_a, I_b, I_c. Prove that in triangle $I_aI_bI_c$, A, B, C are the feet of the altitudes and I is the orthocenter.

11. Let ABC be a triangle with incenter I, and whose incircle is tangent to sides BC, AC, AB at D, E, F, respectively. Let M, N be midpoints of BC, AC, respectively. Prove that EF, BI, MN concur.

12. (Simson Line) Let ABC be a triangle and D a point on its circumcircle. Prove that the feet of the perpendiculars from D to lines AB, AC, and BC are collinear.

13. (Nine Point Circle) Let ABC be a triangle with orthocenter H, altitudes AA_1, BB_1, and CC_1, and midpoints A_2, B_2, C_2. Let the midpoints of AH, BH, CH be A_3, B_3, C_3. Show that $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$ lie on a circle.

2 Problems

1. In parallelogram $ABCD$, let the bisector of $\angle BCD$ intersect lines AB and AD at E and F, respectively. Prove that $BE = AD$ and $DF = AB$.

2. Let ABC be a right triangle, let $\angle C$ be the right angle, and let D be the foot of the altitude from C. Prove that the circumcenters of ACD, CBD, and ABC form a triangle that is similar to ABC.

3. ω_1, ω_2 are two circles intersecting at P and Q. Let A be a variable point on ω_1, and B, C be the intersections of AP, AQ with ω_2. Show that the size of $\widehat{BC}$ is independent of A.

4. (British Math Olympiad 2000) Two intersecting circles C_1, C_2 have a common tangent PQ with P on C_1, Q on C_2. The two circles intersect at M, N, where PQ is nearer to M. The line PN meets the circle C_2 again at R. Prove that MQ bisects $\angle PMR$.

5. ("Largely Artistic Math Olympiad") Two circles ω_1, ω_2 intersect at P, Q. If a line intersects ω_1 at A, B and ω_2 at C, D such that A, B, C, D lie on the line in that order, and P and Q lie on the same side of the line, compute $\angle APC + \angle BQD$.

6. Let P be a point inside circle ω. Consider the set of chords of ω that contain P. Prove that their midpoints all lie on a circle.

7. (British Math Olympiad 2005) The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the line segment AC, and BX meets S again at Q. Show that $BX = CX$ iff PQ is a diameter of S.

8. (Own) In triangle ABC, M, and N are midpoints of AC, and AB. Point D is on BC such that $MD = MC$. Extend lines MD and ND to meet AB, and AC at F, and E, respectively. Prove that EF is perpendicular to BC.

9. (ELMO 2012) In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

10. (USA(J)MO 2010) Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

11. (IMO Shortlist 2010) Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

12. Let $ABCDE$ be a convex pentagon such that $BCDE$ is a square with center O and $\angle A = 90$. Prove that AO bisects $\angle BAE$.

13. (IMO 2006) Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$ and that equality holds if and only if $P = I$.

14. (IMO 2002) The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and F. Prove that I is the incenter of the triangle CEF.

15. (Balkan MO 2012) Let A, B and C be points lying on a circle Γ with centre O. Assume that $\angle ABC > 90$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let l be the line through D which is perpendicular to AO. Let E be the point of intersection of l with the line AC, and let F be the point of intersection of Γ with l that lies between D and E. Prove that the circumcircles of triangles BFE and CFD are tangent at F.

16. (APMO 2007) Let ABC be an acute angled triangle with $\angle BAC = 60°$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

17. In scalene triangle ABC, H, I, and O are the orthocenter, incenter, and circumcenter respectively. Prove one of the angles of the triangle is $60°$ if and only if $IH = IO$.

18. (EGMO 2012) Let ABC be an acute-angled triangle with circumcircle Γ and orthocenter H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM and BC are concurrent.

19. (CGMO 2012) In triangle ABC, $AB = AC$. Point D is the midpoint of side BC. Point E lies outside the triangle ABC such that $CE \perp AB$ and $BE = BD$. Let M be the midpoint of segment BE. Point F lies on the minor arc $\widehat{AD}$ of the circumcircle of triangle ABD such that $MF \perp BE$. Prove that $ED \perp FD$.

20. In triangle ABC, let $D \in BC$, $E \in AC$, $F \in AB$ be the points of tangency of the incircle to the sides. Let I be the incenter. The parallel line through A to BC intersects DE and DF at M and N, respectively. Let L and T be the midpoints of the segments ND and DM. Show that A, L, I, T lie on a circle.

21. (EGMO 2012) Let ABC be a triangle with circumcenter O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. Let K be the circumcenter of triangle AFE. Prove that the lines DK and BC are perpendicular.

22. (IMO 2010) Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the intersection of lines EI and DG lies on Γ.

23. (Romanian TST 1996) Let $ABCD$ be a cyclic quadrilateral and let $\mathcal{M}$ be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (16 points in total). Prove that there are two sets $\mathcal{K}$ and $\mathcal{L}$ of four parallel lines each, such that every line in $\mathcal{K} \cup \mathcal{L}$ contains exactly four points of $\mathcal{M}$.